ON THE ANALYSIS OF THE EFFICIENCY OF A MASS SERVICING SYSTEM

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The choice of parameters in mass servicing systems is commonly based on the average characteristics of the flow of calls. The probability with which the analyzed system will be performing its task is then unknown.

A statement of the problem is set forth below. It is required to determine the minimum possible efficiency of the system which will assure with a given probability that the time spent by any customer coming in during the operation of the system will not exceed a given value. A solution of the problem is obtained for an ordinary flow of independent, standard size calls. An example of the computation is given for the case of a steady flow of calls.

1. Statement of the problem. A single-line mass servicing system operating during time T is considered. The efficiency of the system is characterized by the time τ spent in attending to a standard size call. The system attends to the calls one at a time, in the order of their arrival. On finding the system engaged, an arriving customer takes its place in a queue and waits until the preceding customer has been attended to (the queue is loss-free). The interval of time between the instant of arrival of a customer and the instant of completion is called the time spent by a customer. All the calls are presumed to be of standard size.

In order to characterize the process of arrival of calls, let us introduce the following notation, in accordance with [1]: $v_k(t, \Delta t)$ is the probability of receiving k calls during the interval $(t, t+\Delta t)$ is the probability of receiving k calls during the interval $(t, t+\Delta t)$; $w(t, \Delta t) = 1 - v_0(t, \Delta t)$ is the probability of receiving at least one call during the interval $(t, t+\Delta t)$; $\psi(t, \Delta t) = 1 - v_0(t, \Delta t) - v_1(t, \Delta t)$ is the probability of receiving at least two calls during the interval $(t, t+\Delta t)$. Two hypothesis are adopted regarding the characteristics of the flow of calls [1 and 2].

The first hypotheses relates to the ordinariness of the flow

$$\lim_{\Delta t \to 0} \frac{\psi(t, \Delta t)}{\Delta t} = 0 \tag{1.1}$$

i.e. the probability of receiving at least two calls during an interval of length Δt is infinitely small compared with Δt, when $\Delta t \to 0$.

The second hypothesis is that of independence: for any $t \in (0, T)$ and any $\Delta t > 0$ the probability $v_k(t, \Delta t)$ is independent of the number of incoming calls during the preceding interval $(0, t)$, i.e. the unconditional probability $v_k(t, \Delta t)$ coincides with the conditional probability of receiving k calls during the interval $(t, t+\Delta t)$ for any assumptions regarding the arrival of calls during the interval $(0, t)$.

We assume that the so-called call flow parameter [1 and 2] is known

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{w(t, \Delta t)}{\Delta t} \tag{1.2}$$

According to the above assumptions, $\lambda(t)$ coincides with the intensity of flow (defined as the mathematical expectation of the number of calls per unit time). Therefore, the average number $\langle n \rangle$ of calls during time T will be

$$\langle n \rangle = \int_0^T \lambda(t)\, dt \tag{1.3}$$

The probability of receiving k calls during the interval $(t, t + \Delta t)$ can be expressed in terms of the flow parameter as follows:

$$v_k(t, \Delta t) = \frac{1}{k!} [\Lambda(t + \Delta t) - \Lambda(t)]^k e^{-[\Lambda(t+\Delta t) - \Lambda(t)]} \tag{1.4}$$

$$\left(k = 0, 1, 2, \ldots, \ \Lambda(t) = \int_0^t \lambda(t)\, dt\right)$$

The final formulation of the problem is as follows. The time of the operation of the system T, the intensity of the call flow $\lambda(t)$ when $0 \leq t \leq T$, the maximum permissible call waiting time t_* and the probability R that the service will be completed are given, and we must find the minimum efficiency of the system (maximum time τ) which will ensure with the probability R that no call received during time T will be kept waiting longer than t_*.

2. The expected arrival law. Let us denote by $\nu(t)$ the total number of customers arrived during the interval $(0, t)$ regardless of whether they have been served or not. This is a random function, which can vary from one customer to the next.

We shall now set an intermediary problem relating to the construction of a nonrandom, piece-wise constant nondecreasing function $n(t)$ with integer values n_j

$$n(t) = n_j \quad \text{when } t_j < t < t_{j+1};\ j = 0, 1, \ldots, m \qquad (n_j \leqslant n_{j+1},\ t_0 = 0,\ t_{m+1} = T) \tag{2.1}$$

which will, with the given probability R, limit from above all the occurrences of $\nu(t)$ over the entire time interval $(0, T)$. The choice of the output of the system by this rule, $n(t)$ will ensure with the probability R that all the customers $\nu(t)$ will be served, whilst the customer waiting time will not exceed a given value.

Let us solve the inverse problem. Let the rule (2.1) be given (i.e. the instants t_j, the values $n_j - n_{j-1}$ and the number m of jumps be given). We shall find what is the probability that the inequality

$$\nu(t) \leqslant n(t), \qquad t \in [0, T] \tag{2.2}$$

holds.

Function $n(t)$ is piece-wise constant (2.1), therefore the inequality (2.2) is equivalent to the following system of inequalities

$$\nu(t_j - 0) \leqslant n_{j-1} \qquad (j = 1, \ldots, m + 1;\ n_{j-1} \leqslant n_j) \tag{2.3}$$

Let us denote by A an event, which takes place when the system of inequalities (2.3) holds. Let us also introduce the events $A_{k_0 k_1 \ldots k_m}$ which occur when the system of equalities

$$\nu(t_j - 0) = k_{j-1} \qquad (j = 1, \ldots, m + 1;\ k_{j-1} \leqslant k_j) \tag{2.4}$$

holds.

If two sets of numbers $(k_0, k_1, \ldots, k_n)$ and $(k_0', k_1', \ldots, k_m')$ do not coincide, (i.e. at least one of the equalities $k_j = k_j'$, $j = 0, 1, \ldots, m$ does not hold), then the events $A_{k_0 k_1 \ldots k_m}$ and $A_{k_0' k_1' \ldots k_m}$ are not simultaneous. The event A is represented in the form of the union of these events for the various sets $(k_0, k_1, \ldots, k_m)$ satisfying the inequalities

$$k_j \leqslant n_j,\ k_j \geqslant k_{j-1}\ (j = 0, 1, \ldots, m;\ k_{-1} = 0)$$

$$A = \underset{k_{j-1} < k_j < n_j}{U} A_{k_0 k_1 \ldots k_m} \qquad (j = 0, 1, \ldots, m;\ k_{-1} = 0) \tag{2.5}$$

Hence, the probability of the event A is equal to the sum of probabilities of the events $A_{k_0 k_1 \ldots k_m}$ for all the indicated sets $(k_0, k_1, \ldots, k_m)$

$$P\{A\} = \sum_{k_{j-1} < k_j < n_j} P\{A_{k_0 k_1 \ldots k_m}\} \qquad (j = 0, 1, \ldots, m;\ k_{-1} = 0) \tag{2.6}$$

since the events $A_{k_0 k_1 \ldots k_m}$ are not simultaneous (see, for example, [3]).

We say that event $A_{k_0 k_1 \ldots k_m}$ takes place when $(m + 1)$ independent events occur simultaneously (in accordance with the independence hypothesis)

$$\nu(t_{j+1} - 0) - \nu(t_j + 0) = k_j - k_{j-1} \qquad (j = 0, 1, \ldots, m;\ k_{-1} = 0) \tag{2.7}$$

The probability of each of them is $v_{k_j - k_{j-1}}(t_j, t_{j+1} - t_j)$ and, therefore, by the probability multiplication rule [3]

$$P\{A_{k_0 k_1 \ldots k_m}\} = \prod_{j=0}^{m} v_{k_j - k_{j-1}}(t_j, t_{j+1} - t_j) \tag{2.8}$$

Substituting consecutively (1.4) into (2.8) (when $k = k_j - k_{j-1}$, $t = t_j$, $\Delta t = t_{j+1} - t_j$) and (2.8) into (2.6), we obtain an expression for the required probability

$$P\{A\} = e^{-\langle n \rangle} \sum_{k_m=0}^{\mu_m} \sum_{k_{m-1}=0}^{\mu_{m-1}} \cdots \sum_{k_0=0}^{\mu_0} \prod_{j=0}^{m} \frac{\Delta\Lambda_j^{k_j - k_{j-1}}}{(k_j - k_{j-1})!} =$$

$$= e^{-\langle n \rangle} \sum_{k_0=0}^{n_0} \sum_{k_1=k_0}^{n_1} \cdots \sum_{k_m=k_{m-1}}^{n_m} \prod_{j=0}^{m} \frac{\Delta\Lambda_j^{k_j - k_{j-1}}}{(k_j - k_{j-1})!} \qquad \begin{pmatrix} \mu_0 = \min(n_0, k_1) \\ \cdots\cdots\cdots \\ \mu_{m-1} = \min(n_{m-1}, k_m) \\ \mu_m = n_m \end{pmatrix} \tag{2.9}$$

Here

$$\Delta\Lambda_j = \int_{t_j}^{t_{j+1}} \lambda(t)\, dt, \quad \langle n \rangle = \Lambda(T) = \sum_{j=0}^{m} \Delta\Lambda_j, \quad k_{-1} = 0, \quad t_0 = 0, \quad t_{m+1} = T \tag{2.10}$$

Thus in the subsequent solution of the problem the values t_j, n_j, and m, determining the expected arrival rule $n(t)$, must be chosen that the condition $P\{A\} \geq R$ is satisfied.

3. The minimum efficiency of the system. Up to the time t, $\nu(t)$ calls have arrived; up to that instant the system can attend to t/τ calls (τ being the time required to attend to one call). The call waiting time will not exceed t_* if the solution set of $\nu(t)$ is situated on or below the line

$$l(t) = (t + t_*)/\tau \geqslant \nu(t), \qquad t \in [0, T] \tag{3.1}$$

This is clear from geometrical considerations (Fig. 1). Consequently, only those functions (2.1) which satisfy the condition (3.1) can be considered as the expected rules $n(t)$.

To simplify the process of constructing the expected call arrival rule, let us pass on from the problem of finding the minimum efficiency of the system when the probability of the solution occurring is fixed, to the reciprocal problem. Let us assume the efficiency of the system to be given and seek a rule $n(t)$ which will assure the maximum probability R.

Let us compare the probabilities of occurrence for two rules $n^{(1)}(t)$ and $n^{(2)}(t)$ such that

$$n^{(1)}(t) \geqslant n^{(2)}(t) \qquad t \in [0, T] \tag{3.2}$$

One can at once draw a conclusion with regard to the probabilities of their occurrence

$$P\{n^{(1)}(t) \geqslant \nu(t), \quad t \in [0, T]\} \geqslant P\{n^{(2)}(t) \geqslant \nu(t), \quad t \in [0, T]\} \qquad (3.3)$$

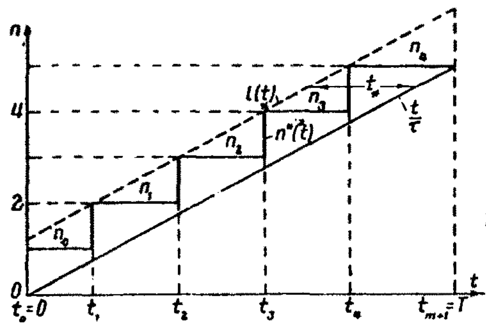

Fig. 1

since all the situations favorable to the second event are also favorable to the first, but there still remain alternatives which are favorable to the first event only.

Thus, the rule $n^*(t)$, assuring the maximum probability of occurrence and satisfying the constraint (3.1) with regard to the waiting time is constructed from the condition (Fig. 1)

$$n^*(t) = \max\{0, 1, 2, \ldots\} \leqslant (t + t_*)/\tau, \qquad n^*(T) \leqslant T/\tau \qquad (3.4)$$

Hence the levels n_j, the instants t_j, and the number m of jumps, determining the expected rule (2.1) will be:

$$n_0 = E(t_*/\tau), \quad n_j = n_0 + j, \quad t_j = (n_0 + j)\tau - t_* \quad (j = 1, \ldots, m),$$

$$m = E(T/\tau) - n_0 \qquad (3.5)$$

Here $E(t_*/\tau)$ and $E(T/\tau)$ denote the integral part of the numbers t_*/τ and T/τ (the largest integer not exceeding the given number).

Substituting (3.5) into (2.9) and (2.10), we obtain an expression (the efficiency of the system, characterized by τ, being given) for the maximum probability of occurrence of the solution

$$R = e^{-\langle n\rangle} \sum_{k_m=0}^{\mu_m} \sum_{k_{m-1}=0}^{\mu_{m-1}} \cdots \sum_{k_0=0}^{\mu_0} \prod_{j=0}^{m} \frac{\Delta\Lambda_j^{\,k_j - k_{j-1}}}{(k_j - k_{j-1})!} \qquad \begin{pmatrix} \mu_0 = \min(n_0, k_1) \\ \cdots\cdots \\ \mu_{m-1} = \min(n_0 + m - 1, k_m) \\ \mu_m = n_0 + m \end{pmatrix}$$

$$\Delta\Lambda_0 = \int_0^{(n_0+1)\tau - t_*} \lambda(t)\,dt, \qquad \Delta\Lambda_m = \int_{(n_0+m)\tau - t_*}^{T} \lambda(t)\,dt \qquad (3.6)$$

$$\Delta\Lambda_j = \int_0^{\tau} \lambda[(n_0 + j)\tau - t_* + \xi]\,d\xi \quad (j = 1, \ldots, m-1), \quad \langle n\rangle = \int_0^T \lambda(t)\,dt$$

$$n_0 = E(t_*/\tau), \quad m = E(T/\tau) - n_0, \quad k_{-1} = 0$$

The derived formulas fully solve the problem, which is the reciprocal of the original one. They can also be regarded as an implicit expression of the minimum efficiency of the system, characterized by the time τ of attending to a single call, by means of the probability of occurrence R in the original problem

4. Stationary arrival flow. Let us consider the case of a stationary arrival flow

$$\lambda(t) \equiv \lambda_0, \qquad t \in [0, T] \qquad (4.1)$$

Then the integrals (3.7) will be:

$$\Delta\Lambda_0 = \Delta\tau_0 / x \qquad (\Delta\tau_0 = 1 + E(t_*/\tau) - t_*/\tau) \qquad (4.2)$$

$$\Delta\Lambda_j = 1/x \qquad (j = 1, \ldots, m-1), \qquad \langle n\rangle = \lambda_0 T$$

$$\Delta\Lambda_m = \Delta\tau_m / x \qquad (\Delta\tau_m = T/\tau - E(T/\tau) + t_*/\tau)$$

Here x denotes the dimensionless efficiency of the system

$$x = \frac{1}{\lambda_0\tau} = \frac{T/\tau}{\lambda_0 T} = \frac{T/\tau}{\langle n\rangle} \qquad (4.3)$$

which is the ratio of the number of calls which can be serviced during time T to the mathematical expectation of the number of calls received during that time.

Substituting relations (4.2) into (3.6) we obtain the following expression for the probability of occurrence of the solution

$$R = e^{-\langle n\rangle} \sum_{k_m=0}^{\mu_m} x^{-k_m} \sum_{k_{m-1}=0}^{\mu_{m-1}} \frac{\Delta\tau_m^{k_m-k_{m-1}}}{(k_m-k_{m-1})!} \sum_{k_{m-2}=0}^{\mu_{m-2}} \frac{1}{(k_{m-1}-k_{m-2})!} \cdots$$

$$\cdots \sum_{k_j=0}^{\mu_j} \frac{1}{(k_{j+1}-k_j)!} \cdots \sum_{k_0=0}^{\mu_0} \frac{1}{(k_1-k_0)!} \frac{\Delta\tau_0^{k_0}}{k_0!} \qquad \left(\begin{array}{l} \mu_0 = \min(n_0, k_1) \\ \cdots\cdots\cdots\cdots \\ \mu_{m-1} = \min(n_0+m-1, k_m) \\ \mu_m = n_0+m \end{array}\right) \tag{4.4}$$

Let us compute the first $(m-1)$ sums in $k_0, \ldots, k_{m-2}$, having written for them the recurrent relations

$$s_0^{(r)} = \frac{\Delta\tau_0^r}{r!}, \quad s_1^{(r)} = \sum_{q=0}^{\min(n_0, r)} \frac{s_0^{(q)}}{(r-q)!}, \ldots, s_{j+1}^{(r)} = \sum_{q=0}^{\min(n_0+j, r)} \frac{s_j^{(q)}}{(r-q)!} \tag{4.5}$$

$$(r = 0, 1, \ldots, n_0+j;\ j = 1, \ldots, m-2)$$

Making use of the method of complete induction we obtain explicit expressions for the sums introduced:

When $r = 0, 1, \ldots, n_0$

$$s_j^{(r)} = \frac{1}{r!} (j + \Delta\tau_0)^r \tag{4.6}$$

When $r = n_0+1, \ldots, \quad n_0+j \qquad (j = 1, \ldots, m-1)$

$$s_j^{(r)} = \frac{1}{r!} (j+\Delta\tau_0)^r - (j+n_0+1-r) \sum_{l=n_0+1}^{r} \frac{(j+n_0+1-l)^{r-l-1}}{(r-l)!} \frac{(l-n_0-1+\Delta\tau_0)^l}{l!}$$

The probability R can, in accordance with (4.4) and (4.5) be written down in the form $(k_m = q,\ k_{m-1} = r)$

$$R = e^{-\langle n\rangle} \sum_{q=0}^{n_0+m} x^{-q} \sum_{r=0}^{\mu} s_{m-1}^{(r)} \frac{\Delta\tau_m^{q-r}}{(q-r)!} \qquad (\mu = \min[n_0+m-1, q]) \tag{4.7}$$

Substituting here the expression for $s_{m-1}^{(r)}$ from (4.6) and summing the positive terms in r, we obtain

$$R = e^{-\langle n\rangle} \left[\sum_{q=0}^{n_0+m} \frac{\langle n\rangle^q}{q!} - \frac{1}{n_0! x^{n_0}} \sum_{i=1}^{m} \frac{1}{x^i} \sum_{j=1}^{i} (m-j) \times \right.$$

$$\left. \times \frac{\Delta\tau_m^{i-j}}{(i-j)!} \sum_{k=1}^{j} \frac{(m-k)^{j-k-1}}{(j-k)!} \frac{(k-1+\Delta\tau_0)^{n_0+k}}{(n_0+k)(n_0+k-1)\ldots(n_0+1)} \right] \tag{4.8}$$

where we assume that $(m-m)(m-m)^{-1} = 1$, for $i = j = k = m$.

If the maximum permissible waiting time is zero ($t_* = 0$, $n_0 = 0$, $\Delta\tau_0 = 1$), the sum in k can be computed

$$(m-j) \sum_{k=1}^{j} \frac{(m-k)^{j-k-1}}{(j-k)!} \frac{k^k}{k!} = \frac{m^{j-1}}{(j-1)!} \tag{4.9}$$

and then also the sum in j

$$\sum_{j=1}^{i} \frac{m^{j-1}}{(j-1)!} \frac{\Delta\tau_m^{i-j}}{(i-j)!} = \frac{(m+\Delta\tau_m)^{i-1}}{(i-1)!} = \frac{(x\langle n\rangle)^{i-1}}{(i-1)!} \tag{4.10}$$

The final expression for the probability in this case is as follows:

$$R = e^{-\langle n\rangle}\left[\frac{\langle n\rangle^m}{m!} + \left(1 - \frac{1}{x}\right)\sum_{q=0}^{m-1}\frac{\langle n\rangle^q}{q!}\right] \quad \text{for } t_* = 0 \tag{4.11}$$

The probability (4.8) is determined by three dimensionless quantities

$$\langle n\rangle = \lambda_0 T_?, \qquad x = 1/(\lambda_0\tau), \qquad \omega = \lambda_0 t_* \tag{4.12}$$

namely, by the mathematical expectation of the number of calls $\langle n\rangle$, the dimensionless efficiency of the system x, and the dimensionless waiting time ω (characterized by the mathematical number of calls during the maximum permissible waiting time). The parameters n_0, $\Delta\tau_0$, m, and $\Delta\tau_m$ in (4.8) can be expressed in terms of $\langle n\rangle$, x and ω (see (3.7) and (4.2))

$$n_0 = E(\omega x), \qquad \Delta\tau_0 = 1 + E(\omega x) - \omega x \tag{4.13}$$

$$m = E(\langle n\rangle x) - E(\omega x), \qquad \Delta\tau_m = \langle n\rangle x - E(\langle n\rangle x) + \omega x$$

For large values of m the oomputations by Formula (4.8) are cumbersome. However, an asymptotic formula can be derived in this instance.

Let us introduce the following notation for the sum in k and the triple sum from (4.8)

$$S_j = (m - j)\sum_{k=1}^{j}\frac{(m-k)^{j-k-1}}{(j-k)!}\,\frac{(k - 1 + \Delta\tau_0)^{n_0+k}}{(n_0+k)(n_0+k-1)\ldots(n_0+1)} \tag{4.14}$$

$$\Omega = \sum_{i=1}^{m}\frac{1}{x^i}\sum_{j=1}^{i} S_j\,\frac{\Delta\tau_m{}^{i-j}}{(i-j)!}$$

then (4.8) can be written in the form

$$R = e^{-\langle n\rangle}\left(\sum_{q=0}^{n_0+m}\frac{\langle n\rangle^q}{q!} - \frac{\Omega}{n_0!\,x^{n_0}}\right) \tag{4.15}$$

Let us change the summation order in the expression (4.14) for Ω and introduce a new summation index $l = i - j$

$$\Omega = \sum_{j=1}^{m}\frac{S_j}{x^j}\sum_{l=0}^{m-j}\frac{1}{l!}\left(\frac{\Delta\tau_m}{x}\right)^l \tag{4.16}$$

Furthermore, in order to simplify the operations let us consider the case $\langle n\rangle x = E(\langle n\rangle x)$, for which $\Delta\tau_m/x = \omega$ and $m = \langle n\rangle x - n_0$ (See (4.13)). We shall express the sum in l in (4.16) in the exponential form (with index ω)

$$\sum_{l=0}^{m-j}\frac{\omega^l}{l!} = \sum_{l=0}^{\infty}\frac{\omega^l}{l!} - \sum_{l=m-j+1}^{\infty}\frac{\omega^l}{l!} = e^{\omega} - R_{m-j}(\omega) \tag{4.17}$$

Here $R_{m-j}(\omega)$ is the remainder term of the expansion, determined by the Lagrange formula as

$$R_{m-j}(\omega) = \frac{\omega^{m-j+1}}{(m-j+1)!}\,e^{\omega\theta_{m-j}} \quad (0 < \theta_{m-j} < 1) \tag{4.18}$$

We shall represent the sum S_j from (4.14) in the powers of $(\langle n\rangle x)$ substituting $m = \langle n\rangle x - n_0$

$$S_j = \alpha_1\frac{(\langle n\rangle x)^{j-1}}{(j-1)!} + \alpha_2\frac{(\langle n\rangle x)^{j-2}}{(j-2)!} + \cdots + \alpha_k\frac{(\langle n\rangle x)^{j-k}}{(j-k)!} + \cdots + \alpha_j \tag{4.19}$$

The coefficients α_k are independent of j and are

$$\alpha_1 = \frac{\Delta\tau_0{}^{n_0+1}}{(n_0+1)}, \qquad \alpha_2 = \frac{(1+\Delta\tau_0)^{n_0+2}}{(n_0+1)(n_0+2)} - \frac{\Delta\tau_0{}^{n_0+1}(n_0+2)}{n_0+1}, \ldots,$$

$$\alpha_k = (n_0 + k) \sum_{l=1}^{k} (-1)^{k-l} \frac{(n_0 + l)^{k-l-1}}{(k-l)!} \frac{(l - 1 + \Delta\tau_0)^{n_0+l}}{(n_0 + 1) \ldots (n_0 + l)} \tag{4.20}$$

$$(k = 1, \ldots, m)$$

Let us substitute Expressions (4.17) and (4.19) into (4.16). Changing the summation order and replacing j by a new summation index $q = j - k$, we obtain

$$\Omega = e^{\omega} \sum_{k=1}^{m} \frac{\alpha_k}{x^k} \sum_{q=0}^{m-k} \frac{\langle n \rangle^q}{q!} - \sum_{k=1}^{m} \frac{\alpha_k}{x^k} \sum_{q=0}^{m-k} R_{m-(k+q)}(\omega) \frac{\langle n \rangle^q}{q!} \tag{4.21}$$

Let us express the first sum in q in (4.21) analogously to (4.17) and (4.18)

$$\sum_{q=0}^{m-k} \frac{\langle n \rangle^q}{q!} = e^{\langle n \rangle} - R_{m-k}(\langle n \rangle) \tag{4.22}$$

Then Expression (4.20) will have the form

$$\Omega = e^{\omega} e^{\langle n \rangle} \sum_{k=1}^{m} \frac{\alpha_k}{x^k} - e^{\omega} \sum_{k=1}^{m} R_{m-k}(\langle n \rangle) \frac{\alpha_k}{x^k} - \sum_{k=1}^{m} \frac{\alpha_k}{x^k} \sum_{q=0}^{m-k} R_{m-(k+q)}(\omega) \frac{\langle n \rangle^q}{q!} \tag{4.23}$$

Substituting (4.23) into (4.15) and expressing the sum in q in (4.15) in its exponential form, we obtain

$$R = 1 - \frac{e^{\omega}}{n_0!\, x^{n_0}} \sum_{k=1}^{m} \frac{\alpha_k}{x^k} + e^{-\langle n \rangle} \left[\frac{e^{\omega}}{n_0!\, x^{n_0}} \sum_{k=1}^{m} R_{m-k}(\langle n \rangle) \frac{\alpha_k}{x^k} + \right.$$

$$\left. + \frac{1}{n_0!\, x^{n_0}} \sum_{k=1}^{m} \frac{\alpha_k}{x^k} \sum_{q=0}^{m-k} R_{m-(k+q)}(\omega) \frac{\langle n \rangle^q}{q!} - R_{n_0+m}(\langle n \rangle) \right] \tag{4.24}$$

Now let $\langle n \rangle \gg 1$ and $x \gg 1$. Then

$$R \approx 1 - \frac{e^{\omega}}{n_0!\, x^{n_0}} \left(\frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} + \ldots \right) \tag{4.25}$$

Here $\alpha_1, \alpha_2, \ldots$ are given by Formulas (4.20).

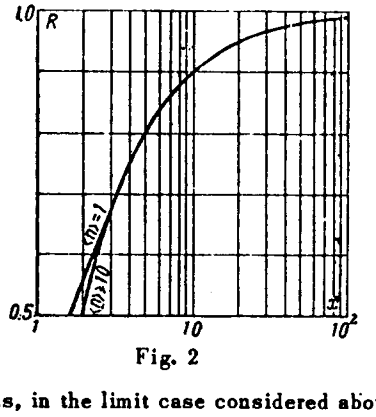

Fig. 2

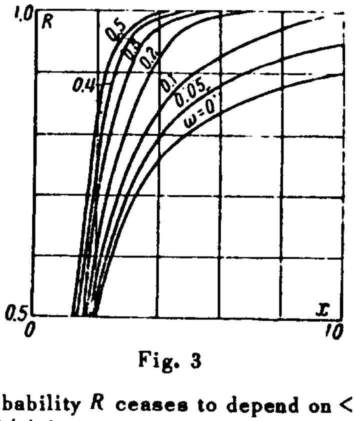

Fig. 3

Thus, in the limit case considered above the probability R ceases to depend on $\langle n \rangle$. This is illustrated in Fig. 2, showing the relation $R(x)$ for various values of $\langle n \rangle$, when $\omega = 0$. The asymptotic formula (4.25) has the form $R = 1 - 1/x$ in this instance.

The effect of the waiting time ω on the probability R can be seen in Fig. 3. When ω increases, and the efficiency x of the system remains unchanged, the probability R increases substantially.

BIBLIOGRAPHY

1. **Khinchin, A.Ia.,** Mathematical Methods in the Theory of Mass Service. Tr. Mat. Inst. im. V.A. Steklov, No. 49, 1955.
2. Gnedenko, B.V., Beliaev, Iu.K. and Solov'ev, A.D., Mathematical Methods in the Theory of Reliability. M., "Nauka", 1965.
3. Gnedenko, B.V., A Course of the Probability Theory. M., "Nauka", 1965.

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